

Computational Exploration of Infectious Disease Dynamics with Limited Healthcare Resources

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Abstract

The presence of limited healthcare resources may greatly influence the spread of the disease and its persistence since the demand for treatment may exceed available healthcare resources. In this study, a mathematical model is presented to understand the dynamics of infectious diseases in a constrained resource environment. The compartmental model is built by dividing the population into compartments of susceptible, infected, hospitalized, and recovered individuals. The healthcare capacity is introduced in the treatment and hospitalization processes of the disease. First, the model is analyzed to prove that its solutions are non-negative and bounded, as well as to find equilibrium points of the disease-free and endemic types. The basic reproduction number is derived in order to know the thresholds of the disease development. After that, local stability conditions are investigated. Results of the analysis are used to show that lack of treatment and hospitalization resources may change the qualitative character of the infectious disease dynamics.

Keywords: Infectious disease; Mathematical modelling; Compartmental model; Healthcare capacity; Disease transmission; Basic reproduction number; Equilibrium analysis; Stability analysis; Epidemic dynamics; Mathematical epidemiology.

1. Introduction

Infectious diseases are still a major threat to public health as their transmission can generate rapidly changing demands for medical treatment, hospitalization and other healthcare services. The evolution of an epidemic is not only related to the biological characteristics of a pathogen and how people interact; it is also related to the capacity of the health system to identify, treat and manage the infectious individuals. When the number of people who need medical help is too high for the health care system to provide, delays in treatment and lack of health care facilities can affect the spread of the disease. The relation between infectious disease dynamics and health care problems is a major problem in mathematical epidemiology. Mathematical modelling is a systematic framework for describing and studying the spread of infectious diseases and the mechanisms of their persistence or elimination. Among the most widely used epidemiological models, compartmental models are the most popular ones because they represent a population in epidemiologically relevant classes, and describe the transitions between them in terms of ordinary differential equations. The classical susceptible-infected-recovered (SIR) model has been the most influential model used to study epidemic transmission in literature and has been extended to include the epidemiological aspects of exposed, hospitalized, vaccinated, treated and other important

compartments. Such models allow important thresholds, equilibrium states, and stability properties to be derived analytically in order to understand the qualitative behaviour of disease transmission. An important assumption in many basic epidemic models is that infected people can be treated or healthcare provided with sufficient resources. Under this assumption, the transition from infection to treatment or recovery can be treated and/or treatment and recovery can be done at a constant rate. However, healthcare systems have finite capacities -- hospitals have limited beds, healthcare facilities have limited treatment capability, and healthcare resources may not be available enough during a period of high transmission. The treatment or hospitalization process may therefore depend on the number of infected people or healthcare resources available. If we ignore that aspect of the epidemic, we may be failing to understand it because the number of infected is so large and medical services are being taken care of by the disease community. In terms of mathematical epidemic models, the health problems can be added as well and they contribute to disease transmission. When healthcare is sufficient, infected people can receive timely treatment which may push them to recovery and reduce the time they contribute to transmissions. In contrast, if healthcare demand is high or healthcare capacity is poor, the effective treatment process can be limited, leading to an increased infected population, longer infection periods, and greater burden on the healthcare system. From a mathematical point of view, this feedback can also have a role in how the infected population is affected by disease development. A good way to illustrate this feedback mechanism is to introduce a hospitalized or treated compartment and allow the transfer from the infected class to the healthcare compartment to depend on the healthcare capacity. Let the total available healthcare capacity be a model parameter. When the number of infected people is still below the effective capacity of treatment, treatment can work well. But as the infected population grows, treatment may be limited. This allows the mathematical analysis to distinguish between disease transmission dynamics that occur with sufficient healthcare availability and those that occur with poor healthcare availability. Healthcare constraints are also closely related to threshold behaviour in epidemic systems and can be investigated in order to understand the threshold behaviour in these systems. The basic reproduction number (R_0) is a basic threshold that describes the expected number of secondary infections that an individual with a typical infectious disease will generate under a given epidemiological condition. In deterministic compartmental models, the R_0 is often a parameter that determines whether disease-free equilibrium has a stable behaviour and is used as a criterion of elimination or recovery. When healthcare capacity is involved in treatment and recovery processes, then it can also influence the threshold of the model. Analytical study of the relationship between healthcare capacity and reproduction threshold can give us information about the conditions under which disease transmission can be controlled. As well as threshold analysis, equilibrium and stability theory can also be a powerful and systematic tool to study the long-term behaviour of the proposed system. The disease-free equilibrium describes the state where no infection exists and the endemic equilibrium is the state for which infection is still present. Finding out if these equilibria exist or are stable is important to know whether a small introduction of infection will disappear or develop into persistent infection in the mathematical model. This analysis is particularly important when healthcare is considered as resource constraints can affect the relationship between infection levels, treatment and recovery. Although mathematical models for infectious disease transmission are considered a prevalent method of analysis in disease management, there is still a need for purely mathematical models that relate the capacity to take care of the diseases with it being well understood. Very detailed mathematical models may contain more than one age group, spatial structure, behaviour change, random effects, vaccination programmes,

delays and pathogen-specific features. Although this extension can be much more realistic, it also introduces much more mathematical complexity. A simpler compartmental model can instead provide a transparent setting where the fundamental impact of limited healthcare resources can be isolated and studied through equilibrium and stability analysis. In this paper, we thus focus on the mathematical investigation of infectious disease dynamics under limited healthcare capacity. We construct a compartmental model in which susceptible, infected, hospitalized or treated and recovered populations are included into the transformation of the infected people to hospitalization or treatment, and from that we model the impact of a finite treatment budget. The resultant system of differential equations is explored analytically to understand its basic mathematical properties and epidemiological thresholds. The main objectives of this work are to develop a mathematically tractable infectious disease model with limited healthcare resources. We first have to establish the positivity and boundedness of the model solutions, and we then obtain the disease-free and endemic equilibrium states and the basic reproduction number and identify the local stability properties of the equilibrium states. In particular, we want to study the mathematics of healthcare capacity and why it is important in spreading disease and its persistence. The main contribution of the work is the development and analysis of a compartmental model in which healthcare capacity is viewed as a part of infectious disease dynamics and not as a separate issue. The study is designed so as to study the relation between transmission, treatment, recovery, and healthcare limitations in a mathematical system, and thus we will understand how resource constraints affect epidemic behaviour. The analytical framework is not dependent on a specific dataset to consider the mathematical mechanisms and thresholds. The rest of the paper is organized as follows. In section 2 the mathematical model is presented as well as the modelling assumptions, compartment definitions, parameters and differential equations. In section 3 we develop the mathematical analysis of the model, positivity and boundedness, equilibrium points, basic reproduction number, and stability conditions. In section 4 we present the main mathematical results and what they mean for infectious disease dynamics under limited healthcare capacity, and conclude with some comments and possible future directions.

2. Mathematical Model

To study the effect of limited health care resources on the dynamics of infectious disease, a deterministic compartmental model is proposed. The total population is divided into susceptible people, infected people, hospitalized or treated people, and recovered people. The model is mathematically tractable yet it accounts for the limited capacity of health care services.

2.1 Model Assumptions

The following assumptions are made in our model:

- The total population is divided into susceptible, infected, hospitalized or treated, and recovered persons.
- Everyone in the susceptible class can get infected by contact with infectious individuals.
- Newly infected individuals enter into the infected compartment.
- Infected individuals can recover naturally or need hospitalization or treatment.
- Hospitalized or treated persons have a higher effective recovery rate than untreated infected people.
- The healthcare system has a finite treatment capacity. If the number of infected persons is relatively small compared to the available healthcare capacity, the infected can get treated. When the number of infected individuals becomes higher and healthcare capacity is not available, the treatment is limited. Recovered people acquire temporary or permanent immunity and they do not return to being susceptible at the time of the model when they are

no longer susceptible. The population is homogeneously mixed such that each susceptible person has an equal chance of contacting an infectious individual. • The demographic processes are the same and there should be a constant recruitment rate and natural death rate. • Disease-induced mortality is included in the infected and/or hospitalized classes. • The model is deterministic and all state variables and parameters are assumed to be non-negative. These assumptions provide a simple description of the relationship between infectious disease and healthcare capacity and avoid the complication of epidemiology.

2.2 Model Formulation

Let

$$S(t), I(t), H(t), R(t)$$

denote, respectively, the numbers of susceptible, infected, hospitalized or treated, and recovered individuals at time t . The total population is therefore

$$N(t) = S(t) + I(t) + H(t) + R(t).$$

Susceptible individuals become infected through contact with infectious individuals. The incidence of infection is represented by the standard bilinear term

$$\frac{\beta SI}{N},$$

where $\beta > 0$ is the effective transmission rate.

Infected individuals leave the infected compartment through natural recovery, hospitalization or treatment, and disease-induced mortality. Let γ denote the natural recovery rate of infected individuals, δ the disease-induced mortality rate, and τ the effective treatment or hospitalization rate. To incorporate limited healthcare resources, the treatment rate is allowed to depend on the available healthcare capacity. Let $K > 0$ represent the effective healthcare capacity. A simple capacity-dependent treatment function can be written as

$$\tau(I) = \frac{\tau_0 K}{K + I},$$

where $\tau_0 > 0$ represents the maximum treatment rate when healthcare resources are readily available.

This formulation has an important mathematical interpretation. When I is small relative to K ,

$$\tau(I) \approx \tau_0,$$

indicating that healthcare resources are sufficiently available. As I increases, the effective treatment rate decreases, reflecting increasing pressure on healthcare resources. Thus, the model does not assume unlimited access to treatment.

Let η denote the recovery rate of hospitalized or treated individuals, μ the natural mortality rate, and δ_H the disease-induced mortality rate among hospitalized individuals. Let Λ denote the recruitment rate into the susceptible population.

The proposed model is given by

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \frac{\beta SI}{N} - \mu S \\ \frac{dI}{dt} &= \frac{\beta SI}{N} - \tau(I)I - \gamma I - (\mu + \delta)I, \\ \frac{dH}{dt} &= \tau(I)I - (\eta + \mu + \delta_H)H, \end{aligned}$$

$$\frac{dR}{dt} = \gamma I + \eta H - \mu R.$$

Using the healthcare-capacity function,

$$\tau(I) = \frac{\tau_0 K}{K + I},$$

the infected population equation becomes

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \frac{\tau_0 KI}{K + I} - (\gamma + \mu + \delta)I.$$

The nonlinear treatment term

$$\frac{\tau_0 KI}{K + I}$$

is the principal mechanism through which limited healthcare capacity enters the model.

For sufficiently small I , the treatment term behaves approximately as

$$\frac{\tau_0 KI}{K + I} \approx \tau_0 I,$$

whereas for large I ,

$$\frac{\tau_0 KI}{K + I} \rightarrow \tau_0 K.$$

Thus, the total treatment flow cannot increase indefinitely with the number of infected individuals. Instead, it approaches the finite upper limit $\tau_0 K$. This property provides a mathematically convenient representation of a healthcare system with limited treatment capacity.

The parameters used in the model are summarized as follows:

Parameter	Description
Λ	Recruitment rate
β	Disease transmission rate
μ	Natural mortality rate
γ	Natural recovery rate of infected individuals
τ_0	Maximum treatment/hospitalization rate
K	Effective healthcare capacity
η	Recovery rate of hospitalized individuals
δ	Disease-induced mortality rate of infected individuals
δ_H	Disease-induced mortality rate of hospitalized individuals

The parameter K is particularly important because it represents the healthcare-resource constraint. A larger value of K corresponds mathematically to greater treatment capacity, whereas a smaller value represents a more constrained healthcare system.

2.3 Fundamental Properties of the Model

The total population satisfies

$$\frac{dN}{dt} = \Lambda - \mu N - \delta I - \delta_H H.$$

Consequently,

$$\frac{dN}{dt} \leq \Lambda - \mu N.$$

By comparison with the corresponding linear differential equation, the total population remains bounded provided that the initial population is finite. In particular,

$$N(t) \leq \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu} \right) e^{-\mu t}.$$

Hence, the biologically feasible region

$$\Omega = \left\{ (S, I, H, R) \in \mathbb{R}_+^4 : N \leq \frac{\Lambda}{\mu} \right\}$$

is positively invariant under the model assumptions.

Because all recruitment, infection, recovery, treatment, and mortality flows are non-negative in the appropriate compartments, solutions beginning with non-negative initial conditions remain non-negative. Therefore,

$$S(t), I(t), H(t), R(t) \geq 0$$

for all $t \geq 0$.

The positivity and boundedness properties ensure that the proposed model is epidemiologically meaningful and mathematically well posed within the biologically feasible region. These properties provide the foundation for the subsequent analysis of equilibrium states, disease transmission thresholds, and stability.

The healthcare-capacity-dependent treatment function is also bounded according to

$$0 < \tau(I) \leq \tau_0,$$

and the total treatment flow satisfies

$$0 \leq \tau(I)I = \frac{\tau_0 KI}{K + I} < \tau_0 K.$$

As a result, this model clearly illustrates that the upper bound in terms of healthcare treatment is finite. This characteristic differentiates the model from the conventional epidemic models that presume limitless availability of treatment as the number of infected individuals rises. The established system creates a mathematically manageable model for exploring the interaction between disease transmission and limited healthcare resources. The following section concentrates on the analytical performance of the system through determining the equilibrium states and computing the basic reproduction number and performing the stability analysis.

3. Mathematical Analysis

In this segment, the qualitative characteristics of the infectious disease model under healthcare capacity limitations are investigated. The analysis examines the disease-free equilibrium, the basic reproduction number, the endemic equilibrium, and stability characteristics of the respective equilibria. The results give models of the conditions in which the disease is expected to die out or to stay in the population.

3.1 Disease-Free Equilibrium

The disease-free equilibrium represents the state of the population in which no infectious or hospitalized individuals are present. At this equilibrium,

$$I = H = R = 0.$$

From the susceptible population equation,

$$\frac{dS}{dt} = \Lambda - \mu S.$$

Setting the derivative equal to zero gives

$$S^0 = \frac{\Lambda}{\mu}.$$

Therefore, the disease-free equilibrium is

$$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0 \right).$$

This equilibrium represents a population in which demographic processes are balanced in the absence of infection.

3.2 Basic Reproduction Number

The basic reproduction number is regarded as a basic threshold parameter in the modeling of infectious diseases. It denotes the anticipated number of secondary infections caused by an average infected person introduced into a healthy population. In order to obtain the reproduction threshold for the chosen model, one should look at the infected compartment close to the disease-free equilibrium E_0 , where the healthcare treatment function fulfils

$$\tau(0) = \tau_0.$$

The linearized infected equation near the disease-free equilibrium therefore becomes

$$\frac{dI}{dt} = [\beta - (\tau_0 + \gamma + \mu + \delta)]I.$$

The infection-generating rate at the disease-free equilibrium is β , while the total rate at which infected individuals leave the infectious compartment is

$$\tau_0 + \gamma + \mu + \delta.$$

Consequently, the basic reproduction number is

$$R_0 = \frac{\beta}{\tau_0 + \gamma + \mu + \delta}.$$

The threshold quantity illustrates the mathematical role of healthcare services. An increase in the effective treatment rate τ_0 increases the rate at which infected individuals leave the infectious class and consequently decreases R_0 . Thus, treatment capacity can influence the transmission threshold of the disease.

The following threshold conditions arise:

$$R_0 < 1$$

indicates that the infection cannot replace itself, whereas

$$R_0 > 1$$

indicates that the infection has the potential to invade the disease-free population.

The critical condition

$$R_0 = 1$$

represents the epidemic threshold separating these two transmission regimes.

It is important to note that R_0 is determined at the disease-free equilibrium, where healthcare resources are not yet under substantial pressure. Therefore, the effect of finite healthcare capacity becomes particularly important when the infected population moves away from the disease-free state.

3.3 Local Stability of the Disease-Free Equilibrium

The local stability of the disease-free equilibrium can be investigated using the Jacobian matrix of the model system.

Let

$$F(S, I, H, R) = (\Lambda - \beta SI - \mu S, \beta SI - (\gamma + \mu + \delta)I - \tau_0 KI, \tau_0 KI - (\eta + \mu + \delta_H)H, \gamma I + \eta H - \mu R).$$

$$\mathcal{F}(S, I, H, R) = \begin{pmatrix} \Lambda - \frac{\beta SI}{N} - \mu S \\ \frac{\beta SI}{N} - \frac{\tau_0 KI}{K + I} - (\gamma + \mu + \delta)I \\ \frac{\tau_0 KI}{K + I} - (\eta + \mu + \delta_H)H \\ \gamma I + \eta H - \mu R \end{pmatrix}.$$

The Jacobian matrix evaluated at the disease-free equilibrium determines the local behaviour of small perturbations around E_0 .

The eigenvalue associated with the infected population is

$$\lambda_I = \beta - (\tau_0 + \gamma + \mu + \delta).$$

Using the definition of R_0 ,

$$\lambda_I = (\tau_0 + \gamma + \mu + \delta)(R_0 - 1).$$

Hence, the sign of the infected eigenvalue is determined directly by $R_0 - 1$.

Therefore, the disease-free equilibrium is locally asymptotically stable when

$$R_0 < 1,$$

and it loses local stability when

$$R_0 > 1.$$

Thus, the basic reproduction number provides the principal threshold governing the initial invasion of infection.

3.4 Endemic Equilibrium

An endemic equilibrium corresponds to a state in which infection remains present in the population.

Denote the endemic equilibrium by

$$E^* = (S^*, I^*, H^*, R^*),$$

where

$$I^* > 0.$$

At equilibrium, all derivatives in the model system vanish. Therefore,

$$\begin{aligned} 0 &= \Lambda - \frac{\beta S^* I^*}{N^*} - \mu S^*, \\ 0 &= \frac{\beta S^* I^*}{N^*} - \frac{\tau_0 K I^*}{K + I^*} - (\gamma + \mu + \delta) I^*, \\ 0 &= \frac{\tau_0 K I^*}{K + I^*} - (\eta + \mu + \delta_H) H^*, \\ 0 &= \gamma I^* + \eta H^* - \mu R^*. \end{aligned}$$

Since $I^* > 0$, the second equilibrium equation can be divided by I^* , giving

$$\frac{\beta S^*}{N^*} = \frac{\tau_0 K}{K + I^*} + \gamma + \mu + \delta.$$

This relationship provides an important interpretation of the endemic state. At equilibrium, the effective force generating new infections must balance the combined removal processes associated with treatment, recovery, and mortality.

From the third equilibrium equation,

$$H^* = \frac{\tau_0 K I^* (K + I^*) (\eta + \mu + \delta_H)}{\tau_0 K I^* (K + I^*) (\eta + \mu + \delta_H)}.$$

Similarly, the recovered population satisfies

$$R^* = \frac{\gamma I^* + \eta H^*}{\mu}.$$

Thus, once the positive infected equilibrium I^* is determined, the corresponding hospitalized and recovered populations follow directly from the equilibrium relations.

The nonlinear healthcare term plays a central role in determining the endemic equilibrium. In particular,

$$\frac{\tau_0 K}{K + I^*}$$

decreases as I^* increases. Consequently, the effective treatment rate per infected individual declines when the infected population becomes large relative to healthcare capacity.

3.5 Role of Healthcare Capacity

The parameter K provides a direct mathematical representation of the available healthcare capacity. The effective treatment flow is

$$\mathcal{T}(I) = \frac{\tau_0 K I}{K + I}.$$

Differentiating this expression with respect to I gives

$$\frac{d\mathcal{T}}{dI} = \frac{\tau_0 K^2}{(K + I)^2} > 0.$$

Thus, total treatment flow increases as the infected population increases. However, the rate of increase becomes progressively smaller as I becomes large. Furthermore,

$$\lim_{I \rightarrow \infty} \mathcal{T}(I) = \tau_0 K.$$

Therefore, the total treatment flow has a finite upper bound. This mathematical property captures the essential effect of limited healthcare resources: treatment availability cannot increase indefinitely with increasing disease burden.

The parameter K also affects the treatment flow directly. For fixed $I > 0$,

$$\frac{\partial \mathcal{T}}{\partial K} = \frac{\tau_0 I^2}{(K + I)^2} > 0.$$

Thus, increased healthcare capacity increases the effective treatment flow for a fixed number of infected individuals. These results illustrate the difference between the disease-free threshold and how the system behaves at higher infection levels. Treatment is close to the maximum rate τ_0 at the disease-free state. As infection increases, healthcare constraints become more important and the treatment process approaches its finite capacity. The mathematical analysis shows that healthcare capacity is not just an external resource variable. Because of the nonlinear treatment mechanism, it directly impacts the relationship between infection prevalence and disease removal. This provides a theoretical basis for studying how finite healthcare resources can contribute to long-term disease transmission. The results

reported in this section provide the basic threshold and equilibrium structure of the model. This is the beginning of the discussion of these mathematical findings and their implications for infectious disease dynamics for different healthcare levels.

4. Discussion and Interpretation

In the earlier sections, the mathematical analysis offers an analytical basis for investigating infectious disease transmission under conditions of limited healthcare availability. In contrast to the compartmental models that assume unlimited treatment availability, the current model uses a capacity-dependent treatment process. It thus creates a mathematical link between the disease burden and the system's treatment resources. The significant conclusion of the analysis is the threshold value of the basic reproduction number. The disease-free equilibrium is said to be locally asymptotically stable for values of $R_0 < 1$ because the model assumes that a small quantity of infection does not result in active transmission. At the same time, in the case of $R_0 > 1$, the disease-free equilibrium becomes unstable and the infected population can spread. Thus, this provides a basic criterion for distinguishing between total disease eradication and potential disease survival. The treatment mechanism adds a further factor to this threshold behaviour. At low rates of infection, the effective treatment rate is already close to its maximum value since health system resources are not used in all cases. But in this case, the infected patients can move relatively quickly from the infected compartment to the hospital or treated compartment. As the number of infected individuals increases, the treatment process becomes increasingly restricted by the finite healthcare capacity. This is a direct consequence of the saturating treatment function that was introduced in the model. The total treatment flow increases with the number of infected individuals, but reaches a finite upper limit where the maximum treatment rate and healthcare capacity are taken into account. Consequently, we cannot continue to increase the number of people receiving treatment as the number of infections increases. This is particularly apparent from a mathematical point of view. If we have a treatment capacity that is unlimited, we may see that the removal of infected people increases proportionally with the number of infected people. The problem here is that such proportional growth is not possible. The further the infected population grows, the more we will see that healthcare services will be under pressure, but the treatment flow will not increase in proportion. The healthcare capacity parameter therefore affects the qualitative behaviour of the epidemic system. For a fixed number of infected people, increasing healthcare capacity increases the effective treatment flow. The residence time of individuals in the infectious class decreases and so does the possibility to continue transmission. However, the treatment mechanism reaches saturation at a lower disease burden and infected individuals remain in the infectious class for much longer. We also show that healthcare capacity and disease transmission should not be considered as independent parts of an epidemic. Disease transmission determines the number of people who need treatment, and healthcare capacity determines how many of them can be removed from the infectious class. So epidemiological pressure and healthcare response are in a feedback relationship. The distinction between maximum treatment rate and total treatment capacity is relevant. The maximum treatment rate is the efficiency with which healthcare services can treat infected people when sufficient resources are available. The healthcare capacity is the scale at which healthcare services are unable to treat people. So at least in theory, to be able to control the epidemic more effectively means to increase treatment efficiency or healthcare capacity. The equilibrium analysis implies that disease persistence is also related to the combination of infection generation and removal, i.e., infection

and removal are the same. At an endemic equilibrium, new infections are balanced by treatment, recovery, and mortality processes. Since treatment is capacity-dependent, this balance changes with the level of infection. As the infected population increases, the effective treatment rate per infected person decreases, which can change the equilibrium structure of the system. Healthcare limitations may be stronger when the disease burden is the highest. When the infection prevalence is low, healthcare capacity is not a major impediment. However, as infection increases, the nonlinear treatment mechanism becomes increasingly important. This means that a healthcare system can be effective during periods of relatively low transmission and becomes increasingly constrained during a major epidemic. From the mathematical modelling point of view, this is an important feature of nonlinear treatment functions. A constant treatment rate is not an adequate way to distinguish between an underutilized healthcare system and a system that is close to its maximum capacity. The capacity-dependent formulation used in this paper is a rather simple way to make the difference without any unnecessary model complexity. The analysis also has implications for the interpretation of the basic reproduction number. Since R_0 is evaluated around the disease-free equilibrium, it captures the initial invasion potential of the infection in conditions where healthcare resources are not yet substantially overloaded. It cannot provide a complete picture of the behaviour of the system at high infection levels. However, when the population is large enough, the nonlinear healthcare constraint becomes very important. We should take this into account when interpreting threshold results in resource-constrained epidemic models. The mathematical framework developed here is quite general. It does not depend on a particular infectious disease, geographical population, or observational dataset. Therefore, the parameters can be interpreted according to the characteristics of a particular disease and healthcare system when the framework is adapted for a specific application. The theoretical structure can also be extended without changing its basic mathematical philosophy. There are several extensions to the theory in the future. Vaccination can be introduced by incorporating an additional vaccinated compartment or a vaccination flow from the susceptible class. Treatment prioritization can be provided by changing the healthcare allocation mechanism. Time-dependent healthcare capacity can be added to represent temporary expansion or contraction of medical resources. Age structure, spatial heterogeneity, behaviour changes, and delayed treatment can also be included as long as the model complexity is considered. The framework could be extended to fractional-order systems to investigate memory effects in disease transmission and treatment. However, the present analysis focuses on a much simpler question that is more fundamental: the mathematical relationship between infectious disease transmission and limited healthcare resources. In combination with a system of compartmental organization and a capacity-dependent treatment mechanism, the model gives a clear representation of the health system's constraints. The threshold and equilibrium analysis shows how the treatment capacity can affect the disease dynamics in a simple way without the need for empirical data, numerical simulations, or computer programming. All in all, the mathematical results emphasize the importance of healthcare capacity as an intrinsic component of infectious disease models. The transmission of infection determines healthcare demand, and healthcare capacity is the factor driving how quickly infected people can be treated and removed from the infectious population. The interaction between these two mechanisms can substantially affect the equilibrium and stability characteristics of the epidemic system. In this way, we provide a mathematical basis to study how infectious disease control can be done in a limited health system.

5. Conclusion and Future Scope

We have developed and analyzed a compartmental mathematical model for infectious disease dynamics when the health system has limited resources. The model consists of susceptible, infected, hospitalized or treated, and recovered populations and introduced a capacity-dependent treatment mechanism to reflect the limited availability of health care services. It represents a mathematical representation of a crucial feature of epidemic systems: the ability of the healthcare system to treat infected people is not unlimited and may be increasingly limited as the number of infected people increases. The mathematical results provided for the basic properties of the proposed model, such as the positivity and boundedness of its solution that the model is still epidemiologically relevant and the population variables are within the biologically feasible region. The disease-free equilibrium was derived and used as the basis for calculating the basic reproduction number and investigating the initial invasion behaviour of the infection. The basic reproduction number was used as a threshold condition for disease transmission. If the reproduction number is lower than 1, the disease-free equilibrium is locally stable, that is, infection cannot survive a small enough introduction. If the reproduction number is higher than 1, the disease-free equilibrium is not locally stable and hence there is the possibility of the infection to survive. This threshold condition is a basic mathematical criterion for determining the conditions for the elimination of a disease and the persistence of a disease. An interesting aspect of our model is that it is explicitly represented in terms of healthcare capacity through nonlinear treatment. The treatment flow is increasing with the number of infected people but is finite and so the availability of treatment will not increase indefinitely with the increase in infection rate. This provides a simple representation of healthcare saturation and is different from traditional models that assume that treatment is available no matter how much disease is present. We show that healthcare capacity can affect the transmission of disease and the removal of disease. When the infection prevalence is low, treatment can be initiated much more effectively because of health care resources. When the infection population is high, the treatment mechanism is more limited. This may result in longer disease spread and can change the equilibrium of the epidemic system. Healthcare capacity must be viewed as an intrinsic part of the dynamics of infectious disease and not as a question of management. Also, the equilibrium analysis can give an indication of the disease persistence. At endemic equilibrium, new infections must be balanced by treatment, recovery, and mortality processes. Because the treatment mechanism depends on the health care system capacity, the equilibrium behaviour is affected not only by transmission and recovery rates, but also by the health care system capacity to deal with the infected individuals. The model thus provides a theoretical link between epidemiological transmission and care system capacity. An important advantage of the proposed framework is that it is purely mathematical. The model does not require data on the disease characteristics of the population, numerical simulations, or programming analysis. The main results are obtained by differential equations, equilibrium analysis, threshold conditions, and stability theory. This makes the framework very plausible for analyzing the qualitative impact of healthcare limitations and is good to study the qualitative aspects of them without adding unnecessary model complexity. Some extensions can be considered in future work. Vaccination could be used to study the negative impact of immunization and limited treatment capacity. A separate treatment compartment could be added for different levels of medical treatment. Time-dependent healthcare capacity might represent temporary expansion of hospital facilities or changes in available medical resources during an epidemic. Age-structured or spatial models could be developed to study differences in disease transmission and healthcare demand between population groups or geographical

regions. Mathematical developments could also consider treatment delays, imperfect treatment, reinfection, behavioural changes, or disease-induced changes in healthcare demand. Fractional-order derivatives could be used to investigate memory effects in infection and treatment processes. Optimal control theory could also be used to determine mathematically good combinations of prevention, treatment, and healthcare capacity interventions. Finally, the mathematical framework outlined in this paper can be viewed as an analytical tool to investigate the dynamics of infectious diseases with limited healthcare resources. The model shows the importance of infection transmission and treatment capacity and how the threshold behaviour and the long-term dynamics of an epidemic can be influenced by the scarce resources in healthcare. A theoretical basis for further mathematical studies of infectious diseases in resource-constrained healthcare systems can be developed from this framework.

REFERENCES:

1. Anderson, R. M., & May, R. M. (1991). *Infectious diseases of humans: Dynamics and control*. Oxford University Press.
2. Bailey, N. T. J. (1975). *The mathematical theory of infectious diseases and its applications* (2nd ed.). Griffin.
3. Brauer, F., van den Driessche, P., & Wu, J. (Eds.). (2008). *Mathematical epidemiology*. Springer. <https://doi.org/10.1007/978-3-540-78911-6>
4. Capasso, V., & Serio, G. (1978). A generalization of the Kermack–McKendrick deterministic epidemic model. *Mathematical Biosciences*, 42(1–2), 43–61. [https://doi.org/10.1016/0025-5564\(78\)90006-8](https://doi.org/10.1016/0025-5564(78)90006-8)
5. Diekmann, O., Heesterbeek, J. A. P., & Metz, J. A. J. (1990). On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations. *Journal of Mathematical Biology*, 28(4), 365–382. <https://doi.org/10.1007/BF00178324>
6. Diekmann, O., Heesterbeek, J. A. P., & Roberts, M. G. (2010). The construction of next-generation matrices for compartmental epidemic models. *Journal of the Royal Society Interface*, 7(47), 873–885. <https://doi.org/10.1098/rsif.2009.0386>
7. Hansen, E., & Day, T. (2011). Optimal control of epidemics with limited resources. *Journal of Mathematical Biology*, 62, 423–451. <https://doi.org/10.1007/s00285-010-0341-0>
8. Heesterbeek, J. A. P. (1990). On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations. *Journal of Mathematical Biology*, 28(4), 365–382. <https://doi.org/10.1007/BF00178324>
9. Hethcote, H. W. (2000). The mathematics of infectious diseases. *SIAM Review*, 42(4), 599–653. <https://doi.org/10.1137/S0036144500371907>
10. van den Driessche, P., & Watmough, J. (2002). Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1–2), 29–48. [https://doi.org/10.1016/S0025-5564\(02\)00108-6](https://doi.org/10.1016/S0025-5564(02)00108-6)
11. Wang, W., & Ruan, S. (2004). Bifurcation in an epidemic model with constant removal rate of the infectives. *Journal of Mathematical Analysis and Applications*, 291(2), 775–793.
12. Wang, W. (2006). Backward bifurcation of an epidemic model with treatment. *Mathematical Biosciences*, 201(1–2), 58–71. <https://doi.org/10.1016/j.mbs.2005.12.022>

13. Zhang, Z., & Liu, X. (2008). Backward bifurcation of an epidemic model with saturated treatment function. *Journal of Mathematical Analysis and Applications*, 348(1), 433–443.
<https://doi.org/10.1016/j.jmaa.2008.07.042>
14. Zhang, Z., & Suo, Y. (2009). Qualitative analysis of a SIR epidemic model with saturated treatment rate. *Nonlinear Analysis: Real World Applications*, 10(4), 1886–1897.
15. Zhou, L., & Fan, M. (2012). Dynamics of an SIR epidemic model with limited medical resources revisited. *Nonlinear Analysis: Real World Applications*, 13(1), 312–324.
<https://doi.org/10.1016/j.nonrwa.2011.07.036>
16. Zhou, Y., Wu, J., & Wu, M. (2013). Optimal isolation strategies of emerging infectious diseases with limited resources. *Mathematical Biosciences and Engineering*, 10(5–6), 1691–1701.
<https://doi.org/10.3934/mbe.2013.10.1691>
17. Zhang, X., & Zhang, X. (2014). Analysis of an SEIR epidemic model with saturated incidence and saturated treatment function. *The Scientific World Journal*, 2014, Article 910421.
<https://doi.org/10.1155/2014/910421>
18. Shan, C., & Zhu, H. (2014). Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds. *Journal of Differential Equations*, 257(5), 1662–1688.
<https://doi.org/10.1016/j.jde.2014.05.030>