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Harmonic Curvature, Harmonic Weyl Tensors and Related Geometric Flows on Complete Riemannian Manifolds

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Abstract:
Let (M^n, g) be a complete Riemannian manifold. Harmonic curvature and harmonic Weyl curvature are important geometric conditions closely related to Einstein manifolds, Codazzi tensors and conformal geometry. A manifold is said to have harmonic curvature if the divergence of the Riemann curvature tensor vanishes, i.e. div $R = 0$. By the contracted second Bianchi identity, this condition is equivalent to the Ricci tensor being a Codazzi tensor. For $n \geq 4$, harmonic Weyl curvature is defined by div $W = 0$, which is equivalent to the vanishing of the Cotton tensor. In this paper, we discuss these curvature conditions on complete Riemannian manifolds and study their relationship with Einstein metrics and Ricci flow. Several basic theorems are proved, including the fact that harmonic curvature implies constant scalar curvature and that every Einstein manifold has harmonic curvature. The behavior of Einstein metrics under Ricci flow is also briefly examined.

Keywords: Harmonic curvature, Weyl tensor, Cotton tensor, Ricci tensor, Einstein manifold and Ricci flow etc.

1. Introduction

Curvature is one of the central concepts in Riemannian geometry. Let (M^n, g) be a smooth Riemannian manifold with Levi-Civita connection ∇. The Riemann curvature tensor *R*, Ricci tensor *Ric*, and scalar curvature *S* describe different aspects of the intrinsic geometry of *M*.

A fundamental curvature condition is

div $R = 0$.

A manifold satisfying this condition is called a manifold with harmonic curvature. This condition is closely connected with the second Bianchi identity and the Codazzi property of the Ricci tensor.

For $n \geq 4$, the Riemann curvature tensor can be decomposed as

$$R = W + \frac{1}{n-2} Ric \backslash owedge\, g - \frac{S}{2(n-1)(n-2)} g \backslash owedge\, g$$

where W denotes the Weyl tensor and \owedge is the Kulkarni–Nomizu product.

The condition

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div $W = 0$

is called harmonic Weyl curvature. It is equivalent to the vanishing of the Cotton tensor.

Geometric flows, particularly the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric$$

provide a dynamic method for studying curvature. The purpose of this paper is to briefly investigate these relationships and establish some fundamental results.

2. Preliminaries

For vector fields (X, Y, Z), the Riemann curvature tensor is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y z - \nabla_Y \nabla_X z - \nabla_{|X,Y|} Z.$$

The Ricci tensor is the contraction

$$R_{ij} = g^{kl} r_{kilj}$$

and the scalar curvature is

$$S = g^{ij} R_{ij}$$

The traceless Ricci tensor is

$$E_{ij} = R_{ij} - \frac{S}{n} g_{ij}$$

For two symmetric (0,2)-tensors A and B, the Kulkarni–Nomizu product is

$$(A \backslash owedge B)_{ijkl} = A_{ik}B_{jl} + A_{jl}B_{ik} - A_{il}B_{jk} - A_{jk}B_{il}$$

3. Harmonic Curvature

Definition 3.1

A Riemannian manifold (M^n, g) is said to have harmonic curvature if

$$div\, R = 0$$

or, in local coordinates,

$$\nabla^l R_{ijkl} = 0$$

The contracted second Bianchi identity gives

$$\nabla^l R_{ijkl} = \nabla_j R_{ik} - \nabla_k R_{ij}$$

This identity gives the following important theorem.

4. Theorem 1: Characterization of Harmonic Curvature

A Riemannian manifold (M^n, g) has harmonic curvature if and only if its Ricci tensor is a Codazzi tensor:

$$div\, R = 0 \Leftrightarrow \nabla_j R_{ik} = \nabla_k R_{ij}$$

Proof

From the contracted second Bianchi identity,

$$\nabla^l R_{ijkl} = \nabla_j R_{ik} - \nabla_k R_{ij}$$

If

$$div\, R = 0$$

then

$$\nabla^l R_{ijkl} = 0$$

Therefore,

$$\nabla_j R_{ik} - \nabla_k R_{ij} = 0$$

which gives

$$\nabla_j R_{ik} = \nabla_k R_{ij}$$

Thus the Ricci tensor is a Codazzi tensor.

Conversely, if

$$\nabla_j R_{ik} = \nabla_k R_{ij}$$

then

$$\nabla^l R_{ijkl} = 0$$

Hence

$$div\, R = 0$$

Therefore,

$$div\, R = 0 \Leftrightarrow \mathrm{Ric\, is\, Codazzi}$$

5. Constant Scalar Curvature

Theorem 2

If (M^n, g) has harmonic curvature, then its scalar curvature is constant.

Proof

Since the curvature is harmonic,

$$\nabla_j R_{ik} = \nabla_k R_{ij}$$

Contract (i) and (k). We obtain

$$\nabla^i R_{ij} = \nabla_j S$$

But the contracted Bianchi identity states

$$\nabla^i R_{ij} = \frac{1}{2} \nabla_j S$$

Consequently,

$$\nabla_j S = \frac{1}{2} \nabla_j S$$

Thus,

$$\frac{1}{2} \nabla_j S = 0$$

and therefore

$$\nabla_j S = 0$$

Hence

S = constant.

6. Harmonic Weyl Tensor

For $n \geq 4$, the Weyl tensor is defined by

$$R = W + \frac{1}{n-2} Ric \backslash owedge\, g - \frac{S}{2(n-1)(n-2)} g \backslash owedge\, g$$

The Weyl tensor is trace-free and represents the conformally invariant part of the Riemann curvature tensor.

Definition 6.1

A manifold has harmonic Weyl curvature if

$$divW = 0$$

The Schouten tensor is

$$P_{ij} = \frac{1}{n-2} \quad R_{ij} - \frac{S}{2(n-1)} g_{ij}$$

The Cotton tensor is

$$C_{ijk} = \nabla_k P_{ij} - \nabla_j P_{ik}$$

For $n \geq 4$,

$$divW = \frac{n-3}{n-2} C.$$

Therefore,

$$divW = 0 \Leftrightarrow C = 0$$

7. Einstein Manifolds

Theorem 3

Every Einstein manifold has harmonic curvature.

Proof

Let

$$Ric = \lambda g$$

For $n \geq 3$, the contracted Bianchi identity implies that λ is constant. Since

$$\nabla g = 0$$

we obtain

$$\nabla Ric = \nabla(\lambda g) = 0$$

Thus,

$$\nabla_j R_{ik} = \nabla_k R_{ij}$$

By Theorem 1,

$$div\, R = 0$$

Therefore, every Einstein manifold has harmonic curvature.

$$Ric = \lambda g \Rightarrow div\, R = 0$$

8. Harmonic Weyl Curvature of Einstein Manifolds

Proposition

Every Einstein manifold of dimension $n \geq 4$ has harmonic Weyl curvature.

Proof

For an Einstein manifold,

$$Ric = \lambda g$$

with λ constant. Hence

$$S = n\lambda$$

is also constant.

Therefore,

$$P_{ij} = \frac{1}{n-2}\lambda - \frac{n\lambda}{2(n-1)} g_{ij}$$

Thus

$$P = cg$$

for a constant (c). Since

$$\nabla g = 0$$

we have

$$\nabla P = 0$$

Consequently,

$$C_{ijk} = \nabla_k P_{ij} - \nabla_j P_{ik} = 0$$

Hence

$$divW = 0$$

Therefore,

$$divW = 0$$

9. Ricci Flow

The Ricci flow is given by

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

It is one of the most important geometric evolution equations.

Under Ricci flow, the scalar curvature satisfies

$$\frac{\partial S}{\partial t} = \Delta S + 2|Ric|^2$$

The Riemann curvature tensor satisfies schematically

$$\frac{\partial Rm}{\partial t} = \Delta Rm + Rm * Rm$$

Thus Ricci flow has a parabolic character and tends to regularize geometric irregularities.

10. Einstein Metrics Under Ricci Flow

Theorem 4

If

$$Ric(g_0) = \lambda g_0$$

then the Ricci flow starting from g_0 is

$$g(t) = (1 - 2\lambda t) g_0$$

Proof

Assume

$$g(t) = c(t) g_0$$

Since $c(t)$ is spatially constant, the Levi-Civita connection does not change. Hence

$$Ric(g(t)) = Ric(g_0) = \lambda g_0$$

The Ricci flow equation gives

$$c'(t) g_0 = -2\lambda g_0$$

Therefore,

$$c'(t) = -2\lambda$$

Using

$$c(0) = 1$$

we obtain

$$c(t) = 1 - 2\lambda t$$

Thus,

$$g(t) = (1 - 2\lambda t) g_0$$

11. Consequences

The preceding theorem gives three important cases:

(i) Positive Einstein constant

If

$$\lambda > 0$$

then

$$g(t) = (1 - 2\lambda t) g_0$$

shrinks and becomes singular at

$$T = \frac{1}{2\lambda}$$

(ii) Ricci-flat case

If

$$\lambda = 0$$

then

$$g(t) = g_0$$

Thus Ricci-flat metrics are stationary solutions of Ricci flow.

(iii) Negative Einstein constant

If

$$\lambda < 0$$

then the metric expands under the flow.

12. Complete Manifolds and Rigidity

Completeness is particularly important for global results. Suppose (M, g) is complete and E is the traceless Ricci tensor,

$$E = Ric - \frac{S}{n} g$$

If one can establish a differential inequality

$$\Delta |E|^2 \geq 2 |\nabla E|^2$$

and if

$$|E|(x) \to 0$$

as

$$d(x, p) \to \infty$$

then the maximum principle can be used to conclude that

$$E = 0$$

Hence

$$Ric = \frac{S}{n} g$$

so the manifold is Einstein.

This illustrates a general principle:

Harmonic curvature + global decay $\Rightarrow$ geometric rigidity

13. Conclusion

Harmonic curvature and harmonic Weyl curvature are important curvature conditions on Riemannian manifolds. The condition

$$div\, R = 0$$

is equivalent to the Ricci tensor being Codazzi and implies that the scalar curvature is constant. In dimensions $n \geq 4$,

$$div W = 0$$

is equivalent to the vanishing of the Cotton tensor.

Einstein manifolds form an important class of examples because

$$Ric = \lambda g$$

implies both harmonic curvature and harmonic Weyl curvature. On complete manifolds, additional decay and curvature assumptions can lead to rigidity and classification results.

The Ricci flow

$$\frac{\partial g}{\partial r} = -2Ric$$

provides a natural dynamic framework for studying these structures. In particular, Einstein metrics evolve self-similarly according to

$$g(t) = (1-2\lambda t)g_0$$

Thus harmonic curvature, harmonic Weyl tensors and geometric flows are closely interconnected subjects that combine differential geometry, tensor analysis and geometric evolution equations.

REFERENCES:

1. Besse, A. L. (1987). Einstein Manifolds. Springer, Berlin.
2. Derdziński, A. (1981). On compact Riemannian manifolds with harmonic curvature. Mathematische Annalen, 259, 145–152.
3. Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature. Journal of Differential Geometry, 17, 255–306.
4. Hamilton, R. S. (1995). The formation of singularities in the Ricci flow. Surveys in Differential Geometry, 2, 7–136.
5. Petersen, P. (2016). Riemannian Geometry, 3rd ed. Springer, New York.
6. Chow, B., Lu, P., Ni, L., & Hamilton, R. S. (2006). Hamilton's Ricci Flow. American Mathematical Society.
7. Hebey, E. (1996). Sobolev Spaces on Riemannian Manifolds. Springer.
8. Kobayashi, S., & Nomizu, K. (1963). Foundations of Differential Geometry, Vol. I. Interscience Publishers.
9. Cao, H.-D. (1996). Existence of gradient Kähler-Ricci solitons. In Elliptic and Parabolic Methods in Geometry, 1–16.
10. Catino, G. (2012). Complete gradient shrinking Ricci solitons with pinched curvature. Mathematical Research Letters, 19, 1421–1430.